Eddie Price - MA 266, Challenge Problems (SP 19)

No challenge problems for lesson 1.

Lesson 2.

An *operator* on the set of infinitely differentiable functions (functions for which the nth derivative exists for every positive integer n) is a function which takes infinitely differentiable functions as inputs and also has infinitely differentiable functions as outputs. i.e., if $\mathcal{O}$ is an operator on the set of infinitely differentiable functions, then for any infinitely differentiable function $f(x)$, $\mathcal{O}[f(x)]$ is an infinitely differentiable function. A *linear operator* is an operator L which satisfies the conditions that
1) for every $f(x)$ and $g(x)$, it is the case that $L[f(x) + g(x)] = L[f(x)] + L[g(x)]$, and
2) for every constant c (an actual number) and for every $f(x)$, it is the case that $L[c \cdot f(x)] = c \cdot L[f(x)]$.
(One example of a linear operator is differentiation: $\dfrac{d}{dx}$ since $\dfrac{d}{dx}[f(x) + g(x)] = \dfrac{d}{dx}[f(x)] + \dfrac{d}{dx}[g(x)]$ and $\dfrac{d}{dx}[cf(x)] = c\dfrac{d}{dx}[f(x)]$. Another example is integration where we choose the arbitrary constant C to be 0.)
In class, we saw that an nth order linear differential equation can be written in the form $f_0(x)\, y^{(0)} + f_1(x)\, y^{(1)} + \ldots + f_n(x)\, y^{(n)} = g(x)$ for some functions $f_0(x)\,, \ldots, f_n(x)\,, g(x)$.
Show that the left hand side of the linear differential equation above is a linear operator on the set of infinitely differentiable functions. Also, show that any differential equation which *cannot* be written in the above form of a linear diff eq also cannot be written in the form $L[y] = g(x)$ for any linear operator L and any function $g(x)$. i.e., show that a differential equation is linear if and only if it can be written as $L[y] = g(x)$ for some linear operator L and some function $g(x)$.

Lesson 3.

Consider the first order linear diff eq $\dfrac{dy}{dt} + p(t)\, y = g(t)$. Explain what an integrating factor is for this diff eq. (Explain what it does.) Then show that $\mu(t) = \exp\big(\int p(t)\ dt\big)$ does what you said an integrating factor does.

Lesson 4.

4.1. Recall that a first order diff eq is separable if it can be written in the form $f(x) + g(y)\, \dfrac{dy}{dx} = 0$ for some function $f(x)$ in the variable x and some function $g(y)$ in the variable y. Informally, we "separate" the variables and integrate, but this isn't horribly mathematically valid. I stated that this method could be made valid by seeing that when we "separate" the variables, we get $F(x) + G(y) = C$ (where $\dfrac{dF}{dx} = f(x)$ and $\dfrac{dG}{dy} = g(y)$ and C is an arbitrary constant), and that when we differentiate this equation, we get the differential equation. Explain why this shows that the method of "separating variables" works.

4.2.a. A function $y = f(x)$ is a *solution* to a first order differential equation $\frac{dy}{dx} = g(x, y)$ if plugging in $f(x)$ for y in the differential equation yields a true statement for every value in the domain of y. Explain why we must restrict the domain of y to places where $\frac{dy}{dx}$ is defined.
4.2.b. Consider the first order linear differential equation $h(t)\,\frac{dy}{dt} + p(t)\,y = g(t)$. Let c be a number so that $h(c) = 0$. Explain why c is *not* in the domain of any solution to this diff eq.

Lesson 5.

5. Refer to a first-order differential equation as *bihomogeneous* if there exist positive integers m and n which are relatively prime (i.e., $\gcd(m, n) = 1$) so that the differential equation can be written in the form $\frac{dy}{dx} = g\left(\frac{y^m}{x^n}\right)$ for some function g in the "variable" $\frac{y^m}{x^n}$.

5.a. Consider the substitution $u(x) = \frac{y^m}{x^n}$. Show that the substitution will work if and only if $m = 1$.
5.b. In the case $m = 1$, prove that the differential equation you obtain after your substitution will be separable if and only if $n = 1$ as well (i.e., only in the case where the differential equation is actually homogeneous).
5.c. In the case $m = 1$, describe the symmetries of the direction fields of bihomogenous diff eqs. (in other words, for each positive integer n, describe the symmetry of the direction field.)

No challenge problems for lesson 6 or 7.

Lesson 8.

8. Consider the first order linear differential equation $\frac{dy}{dt} + p(t)\,y = g(t)$, where $p(t)$ and $g(t)$ are continuous for all t-values. Now consider the IVP consisting of the above diff eq and the initial condition $y(0) = y_0$. The method of integrating factors guarantees that a unique solution to the IVP exists. Explain why the solution must be unique. (This is tricky! There are actually two parts to this, not just one! Hint: not only must you show that C is uniquely determined by the initial condition, you must show that there is no other vastly different function that can solve the diff eq. To do this, use one of the implications of the Mean Value Theorem: Any two antiderivatives of the same function differ only by a constant.)

Lesson 9.

Consider the autonomous diff eq $\frac{dy}{dt} = y\sin(\pi y)$. Show that the equilibrium solutions of this diff eq are precisely the equations of the form $y = n$ for an integer n (show every equilibrium solution is of this form and every equation of this form is an equilibrium solution). For each integer n, classify whether $y = n$ is asymptotically stable, unstable, or semistable. Do this without the aid of technology.

Lesson 10.

10. Sometimes we have a differential equation $M(x,y)+N(x,y)\,y'=0$ which is not exact. In this case, it may be possible to use an integrating factor to turn this equation into an exact equation.
10.a. Show that if there is a function $\mu(x,y)$ so that $\mu(x,y)\,M(x,y)+\mu(x,y)\,N(x,y)\,y'=0$ is exact, then μ must satisfy the partial differential equation $M\mu_y - N\mu_x + (M_y - N_x)\,\mu = 0$.
10.b. Now, show that any implicit solution $\psi(x,y)=C$ to the exact differential equation $\mu(x,y)\,M(x,y)+\mu(x,y)\,N(x,y)\,y'=0$ is also a solution to the original differential equation $M(x,y)+N(x,y)\,y'=0$. (Hint: Factor $\mu(x,y)$ out of the exact diff eq.)
10.c. Partial differential equations can be quite difficult to solve, so we may seek an integrating factor $\mu(x)$ which depends only on x. In such a case, $\mu_x = \dfrac{d\mu}{dx}$ and $\mu_y = 0$. From your work in part (a), show that if an integrating factor depending only on x exists, then it must satisfy the first order linear diff eq $\dfrac{d\mu}{dx} + \dfrac{N_x - M_y}{N}\mu = 0$.
10.d. The non-exact diff eq $(3xy+y^2)+(x^2+xy)\,y'=0$ has an integrating factor depending only on x. Find this integrating factor.
10.e. Using the integrating factor from part (d), find an implicit solution to the non-exact diff eq $(3xy+y^2)+(x^2+xy)\,y'=0$.

Lessons 11 and 12.
Consider the IVP $\dfrac{dy}{dx} = \dfrac{1}{3\sqrt[3]{x^2}}$, $y(-1)=-1$. Try to use the online Euler's method calculator to estimate the value of $y(1)$. What happens? Explain why Euler's method does not produce an accurate estimate of $y(1)$.

Lessons 13, 14. No challenge problems

Lesson 15.
15.1. Prove that $\sin(2t)=2\sin(t)\cos(t)$ and $\cos(2t)=\cos^2(t)-\sin^2(t)$ for all real numbers t. (Hint: $e^{2ti}=\left(e^{ti}\right)^2$.)
15.2. Prove that $\sinh(it)=i\sin(t)$ and $\cosh(it)=\cos(t)$ for all real numbers t. From here, conclude that

$$\sin(t)=\frac{e^{it}-e^{-it}}{2i} \qquad \text{and} \qquad \cos(t)=\frac{e^{it}+e^{-it}}{2}$$

Lesson 16. No challenge problems

Lesson 17.
17.1.a. Consider the diff eq $y''-4y=\sin t$. Assume that a particular solution is $Y(t)=A\sin(t)$. Find the value of A which makes the statement true.
17.1.b. Consider the diff eq $y''+3y'-4y=\sin t$. Show that $Y(t)=A\sin(t)$ cannot be the form of the particular solution because, if you assume it is, you get a system of 2 different equations in 1 unknown (the unknown being A), so A would have to be two different numbers simultaneously.

17.1.c. Explain why the existence of a y' term in 17.1.b forces us to add $B\cos(t)$ to our particular solution.

17.2.a. Consider the diff eq $y''+2y'+y=t^2$. One might initially guess that the particular solution would be of the form $Y(t)=At^2$. Explain why the existence of the y' term requires us to add Bt and the existence of the y'' term requires us to add C, giving $Y(t)=At^2+Bt+C$. How is the y term related to the At^2 term in $Y(t)$?
17.2.b. Assume you have a homogeneous second order diff eq with constant coefficents, i.e., $ay''+by'+cy=0$. Suppose that the solution has a constant term. (This happens if one of the roots of the characteristic polynomial is 0.) Show that c must be equal to 0 in this case. (i.e., show that if the solution has a constant term, then the diff eq is of the form $ay''+by'=0$.)
17.2.c. Assume you have a diff eq $y''+y'=t^2$. Our initial guess for $Y(t)$ is At^2+Bt+C (as seen in 17.2.a). Try plugging in this differential equation. What goes wrong?
17.2.d. Explain why multiplying our initial guess for $Y(t)$ by t fixes the issue. (Think about your answers to 17.2.a and 17.2.c).

Lessons 18-21. No challenge problems.

Lesson 22.
22.1. Suppose you have an nth order linear diff eq with constant coefficients

$$a_ny^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_1y'+a_0y=0$$

Assume that you have a solution of the form $y(t)=e^{rt}$ for some constant r. Show that r must satisfy the equation

$$a_nr^n+a_{n-1}r^{n-1}+\ldots+a_1r+a_0=0$$

22.2. Suppose you have a 3rd order diff eq $ay'''+by''+cy'+dy=0$, and suppose you know that y_1, y_2, and y_3 are solutions. The Wronskian of y_1, y_2, and y_3 is the determinant

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = y_1\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}$$

Similarly to 2nd order equations, the set of solutions is a fundamental set if and only if the Wronskian is nonzero.
22.2.a. Show that the set $\{t^2+1, t^2-t, 2t+2\}$ has Wronskian equal to 0.
22.2.b. Show that the set $\{t^2+1, t^2+t, 2t+2\}$ has nonzero Wronskian.

Lesson 23.
23.1.a. Make sure you have done challenge problem 17.1. Suppose you have a diff eq of the form $a_6y^{(6)}+a_4y^{(4)}+a_2y''+a_0y=\sin(t)$. Explain why $Y(t)=A\sin(t)$ works for the particular solution (even though we would normally use $A\sin(t)+B\cos(t)$). (How are odd order derivatives related to $A\sin(t)$ and/or $B\cos(t)$? What about even order derivatives?)
23.1.b. Make sure you have done challenge problem 17.1. Suppose you have a diff eq of the form $a_5y^{(5)}+a_3y'''+a_1y'=\sin(t)$. Explain why $Y(t)=B\cos(t)$ works for the particular

solution (even though we would normally use $A\sin(t) + B\cos(t)$). (How are odd order derivatives related to $A\sin(t)$ and/or $B\cos(t)$? What about even order derivatives?)

23.2.a. Make sure you have done challenge problem 17.2. Suppose you have t^n. If n is odd, show that every odd order derivative is of the form At^m where m is even and that every even order derivative is of the form At^k where k is odd. If n is even, show that every odd order derivative is of the form At^k where k is odd and that every even order derivative is of the form At^m where m is even.
23.2.b. Make sure you have done challenge problem 17.2. Suppose you have a diff eq of the form $a_6y^{(6)} + a_4y^{(4)} + a_2y'' + a_0y = t^4$. Explain why $Y(t) = At^4 + Ct^2 + E$ works for the particular solution (even though we would normally use $At^4 + Bt^3 + Ct^2 + Dt + E$).
23.2.c. Make sure you have done challenge problem 17.2. Suppose you have a diff eq of the form $a_5y^{(5)} + a_3y''' + a_1y' = t^4$. Explain why $Y(t) = At^5 + Ct^3 + Et$ works for the particular solution (even though we would normally use $t(At^4 + Bt^3 + Ct^2 + Dt + E)$ - note, we need to multiply by t because 0 is a root of the characteristic polynomial since the y term is missing from the diff eq).

Lesson 24. No challenge problems.

Lesson 25. We will prove that $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$, where $Y(s) = \mathcal{L}\{y(t)\}$.
25.a. By definition, $\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st}y'(t)\,dt$. Use integration by parts with $u = e^{-st}$ and $dv = y'(t)\,dt$ to show the integral evaluates to

$$e^{-st}y(t)\Big|_0^\infty + \int_0^\infty se^{-st}y(t)\,dt$$

25.b. The integral there is $sY(s)$. Now, the term on the left can be evaluated as

$$\lim_{b\to\infty}\left[e^{-st}y(t)\right]_0^b$$

Understanding L'Hôpital's Rule, one can show that if $y(t)$ is a polynomial or a sinusoidal function, then $\lim_{b\to\infty} e^{-sb}y(b) = 0$. The only other types of functions we care about are of the form $y(t) = e^{ct}$. Show that for $s > c$, we get a limit of 0 here too. Finally, conclude we actually get $sY(s) - y(0)$.
25.c. Now, show that $\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$. (Hint: Notice that $y''(t)$ is the derivative of $y'(t)$, so apply the result twice.)

Lessons 26, 27, 28. No challenge problems.

Lesson 29. An integro-differential equation is an equation involving derivative(s) and integral(s) of a function.
29.a. Solve the integro-differential equation below by differentiating to obtain a differential equation. (Hint: the Fundamental Theorem of Calculus implies that $\frac{d}{dt}\int_a^t f(x)\;dx = f(t)$)

$$y'(t) + \int_0^t (t-\tau)\,y(\tau)\;d\tau = t, y(0) = 0$$

29.b. Now solve the integro-differential equation in 29.a by using the Laplace transform.

Lesson 30. Solve the following system of 2nd order linear diff eqs. (Hint: you can convert the system of 2nd order linear diff eqs into a single 4th order diff eq and solve for one of the functions that way.)

$$\begin{cases} x'' = y \\ y'' = x \end{cases}$$

Lesson 31.
31.a. Consider the system of differential equations

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases}$$

Find the characteristic polynomial of coefficient matrix for this system.
31.b. Convert the above system into a 2nd order linear diff eq in the variable x. What is the characteristic polynomial for the second order equation you got? How does it compare to the characteristic polynomial you got in part a?
31.c. Convert the above system into a 2nd order linear diff eq in the variable y. What is the characteristic polynomial for the second order equation you got? How does it compare to the characteristic polynomial you got in part a and in part b?
31.d. In light of your answers to the previous parts, explain why it isn't surprising that a system of 2 first-order linear diff eqs behaves similarly to a single 2nd order diff eq.

Lesson 32.
32.1.a. Solve the system $\mathbf{x}' = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \mathbf{x}$
32.1.b. Use pplane to sketch several trajectories of the above system. Describe the phase portrait.
32.2.a Solve the system $\mathbf{x}' = \begin{pmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{pmatrix} \mathbf{x}$
32.2.b. Use pplane to sketch several trajectories of the above system. Describe the phase portrait.
32.3.a. Find the eigenvalues of and use pplane to sketch several trajectories of the system $\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}$. Describe the phase portrait.
32.4. What do you hypothesize the phase portrait looks like if 0 is an eigenvalue?

Lesson 33.
33.1. The purpose of this problem is to recognize that "center " points in the phase planes are transitions between asymptotically stable spiral points and unstable spiral points. Philosophically, then, it makes sense that a "center" point is merely stable - it is right in between an inward pull to the origin and an outward pull away from the origin. In other words, we can think of a "center" as having a balanced amount of inward pull and outward pull. We will see this by varying coefficients of the characteristic polynomial for a system.

33.1.a. Suppose your characteristic polynomial is $\lambda^2 + b\lambda + c$ (notice that by the way the characteristic polynomial is defined, we can always get a characteristic polynomial in which the leading coefficient is 1). Notice that, in order to have non-real complex roots, it must be the case that $b^2 < 4c$. Convince yourself that you get a center if and only if $b = 0$ and $c > 0$.
33.1.b. We will fix $c > 0$ and let b vary here. Since we get a center if and only if $b = 0$ in this case, consider values of b slightly less than 0. What happens in the phase portrait in this case? Also, consider values of b slightly greater than 0. What happens in the phase portrait in this case? Conclude that, by fixing $c > 0$ and letting b pass through 0, we get that a center point is a transition between asymptotically stable spiral points and unstable spiral points.
33.1.c. We will fix $b = 0$ here and let c vary. In other words, the characteristic polynomial is $\lambda^2 + c$. For what values of c do we get a center? For what values of c do we get a spiral point? Explain why the qualitative transition of the phase plane in this case is unrelated to spiral points. Explain, then, how centers are related to saddle points, and how one can transition between the two (what happens to the elliptical orbits as c gets closer and closer to 0 (and c is positive)? What do solutions look like when $c = 0$? What happens to trajectories as c gets closer and closer to 0 (and c is negative)?)

Lesson 34.
34.1.a. Solve the system $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{x}$ (Do this the way we have been doing in class by finding eigenvalues and eigenvectors, but also take note that you really are getting the system $x' = 2x$ and $y' = 2y$, which implies that x and y really don't have any relation to each other at all.)
34.1.b. Use pplane to sketch several trajectories of the above system. Describe the phase portrait.
34.1.c. We know that eigenvectors give us linear trajectories (along the eigenvector). Use this fact to explain why the phase portrait looks the way it does.
34.1.d. A fact from linear algebra tells us that if a 2×2 matrix has 2 linearly independent eigenvectors, then the matrix is diagonalizable. In other words, the matrix is a matrix conjugate of a diagonal matrix. In other words, suppose A is a 2×2 matrix with eigenvalues λ_1 and λ_2 with eigenvectors $\mathbf{z}^{(1)} = \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix}$ and $\mathbf{z}^{(2)} = \begin{pmatrix} z_1^{(2)} \\ z_2^{(2)} \end{pmatrix}$, respectively. Then

$$A = \begin{pmatrix} z_1^{(1)} & z_1^{(2)} \\ z_2^{(1)} & z_2^{(2)} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} z_1^{(1)} & z_1^{(2)} \\ z_2^{(1)} & z_2^{(2)} \end{pmatrix}^{-1}$$

Use this fact to show that if A is a 2×2 scalar matrix with repeated real eigenvalues (i.e., $\lambda_1 = \lambda_2$) and if A has 2 linearly independent eigenvectors, then A is a scalar matrix, i.e., $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ for some scalar a. Conclude that if you have a system of diff eqs $\mathbf{x}' = A\mathbf{x}$ in which A has a repeated (nonzero) eigenvalue, then the solutions of the system/phase portrait looks like that in 34.1.a or else has the form we discussed in class with generalized eigenvectors η where the phase portrait has an improper node at the origin. Note then that if the functions of your system affect each other and your scalar matrix has a repeated eigenvalue, then you must use the generalized eigenvector process.

34.2. We will see in this problem how improper nodes are transitions between two other types of phase portrait structure. We may assume (as in 33.1) that the characteristic polynomial is of the form $\lambda^2 + b\lambda + c$. Explain why we get an improper node if and only if $b^2 = 4c$ (we may assume that both b and c are nonzero). If $b^2 < 4c$, what does the phase portrait look like? If $b^2 > 4c$, what does the phase portrait look like? Describe how an improper node is a transition between two phase portrait behaviors, and state which they are.

Lesson 35. Here, we will develop the method of variation of parameters for 1st order linear systems. Suppose you have the system

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$$

35.a. Suppose $\lambda_1, \mathbf{z}^{(1)} = \begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix}$ and $\lambda_2, \mathbf{z}^{(2)} = \begin{pmatrix} z_1^{(2)} \\ z_2^{(2)} \end{pmatrix}$ are eigenvalue/vector pairs for A. Then the "fundamental" matrix $\Phi(t)$ is

$$\Phi(t) = \begin{pmatrix} z_1^{(1)} e^{\lambda_1 t} & z_1^{(2)} e^{\lambda_2 t} \\ z_2^{(1)} e^{\lambda_1 t} & z_2^{(2)} e^{\lambda_2 t} \end{pmatrix}$$

Show that the matrix product $A\Phi(t) = \Phi'(t)$ (hint: use the definition of eigenvalues and eigenvectors - that $A\mathbf{z} = \lambda\mathbf{z}$).

35.b. Show that if $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where c_1, c_2 are arbitrary constants, then $\mathbf{x} = \Phi(t)\mathbf{c}$ is the general solution to the homogeneous system $\mathbf{x}' = A\mathbf{x}$. (Hint: plug it in and use part a.)

35.c. Assume that the general solution to the above system is of the form $\mathbf{x}(t) = \Phi(t)\mathbf{u}(t)$ for some vector $\mathbf{u}(t)$ whose entries are functions of t. (Note how this is similar to variation of parameters for 2nd order equations - we take the complementary solution and let the parameters vary as functions of t rather than being arbitrary constants.) Show that, after making this assumption, it must be the case that $\Phi(t)\mathbf{u}'(t) = \mathbf{g}(t)$. (Hint: plug $\Phi(t)\mathbf{u}(t)$ into the system and use part a.)

35.d. From part c, we can then solve for $\mathbf{u}(t)$ by taking the system of equations from part c, solving for u_1' and u_2' and integrating. Convince yourself that this is true.

35.e. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

using variation of parameters.